AIAA JOURNAL
Vol. 40, No. 2, February 2002

Efficient Method for Dynamic Condensation of Nonclassically Damped Vibration Systems

Zu-Qing Qu*
Shanghai Jiao Tong University, Shanghai 200030, People's Republic of China
and
R. Panneer Selvam†
University of Arkansas, Fayetteville, Arkansas 72701

Dynamic condensation methods have been widely used to reduce the number of degrees of freedom of finite element models. Most of them, however, are valid for undamped systems. An efficient iterative approach for the dynamic condensation of nonclassically damped systems is proposed. The classical subspace iteration method for undamped models is extended to nonclassically damped models. Then, a governing equation for the dynamic condensation matrix in state space is derived from the extended subspace iteration. Two iterative schemes are proposed to solve the governing equation. Because the dynamic condensation matrix is independent of the system matrices and the eigenpairs (eigenvalues and eigenvectors) of the reduced model, it is unnecessary to compute them in every iteration. This makes the present method much more computationally efficient than those approaches proposed in the past. The convergence of the proposed approach is also proven. Two numerical examples, one discrete mass–damper–spring system and one floating raft isolation system, are included to demonstrate the convergence of the present method. The results show that the convergence of the present method is much faster than the previous approaches, especially when the dynamic characteristics of the reduced model are very close to the full model.

Nomenclature

$\boldsymbol{A}$	=	$(2n \times 2n)$ system matrix of the full model in state space defined in Eq. (3)
$\boldsymbol{A}_R$	=	$(2m \times 2m)$ system matrix of the reduced model in state space defined in Eq. (15a)
$\boldsymbol{B}$	=	$(2n \times 2n)$ system matrix of the full model in state space defined in Eq. (3)
$\boldsymbol{B}_R$	=	$(2m \times 2m)$ system matrix of the reduced model in state space defined in Eq. (15b)
$\boldsymbol{C}$	=	$(n \times n)$ damping matrix of the damped system
$\boldsymbol{I}$	=	unity matrix
$\boldsymbol{K}$	=	$(n \times n)$ stiffness matrix of the damped system
k	=	integer, $k > 1$
$\boldsymbol{M}$	=	$(n \times n)$ mass matrix of the damped system
m	=	number of low eigenpairs to be considered; number of the master degrees of freedom
n	=	number of the total degrees of freedom of the full model
$\tilde{\boldsymbol{Q}}$	=	$(2m \times 2m)$ eigenvector matrix of the projected model defined in Eq. (22)
$\boldsymbol{R}$	=	$(2s \times 2m)$ dynamic condensation matrix
s	=	number of the slave degrees of freedom
$\boldsymbol{X}$	=	displacement response vector; new subspace defined in Eq. (20)
$\dot{\boldsymbol{X}}$	=	velocity response vector
$\ddot{\boldsymbol{X}}$	=	acceleration response vector
ε	=	error tolerance of eigenvalue used in Eq. (24)
ε_1	=	error tolerance of the real and imaginary parts of the complex eigenvalue used in Eqs. (42) and (44)
ε_2	=	error tolerance of the column vector in the dynamic condensation matrix used in Eq. (45)
$\tilde{\boldsymbol{\Lambda}}$	=	$(2m \times 2m)$ eigenvalue or spectral matrix of the projected model defined in Eq. (22)
$\lambda_j^{(i)}$	=	the ith approximation of the jth eigenvalue
$\boldsymbol{\Psi}$	=	$(n \times n)$ submatrix of the eigenvector matrix defined in Eq. (4)
$\tilde{\boldsymbol{\Psi}}$	=	$(2n \times 2n)$ eigenvector matrix defined in Eqs. (2) and (4)
$\boldsymbol{\Omega}$	=	$(n \times n)$ submatrix of the eigenvalue or spectral matrix defined in Eq. (4)
$\tilde{\boldsymbol{\Omega}}$	=	$(2n \times 2n)$ eigenvalue or spectral matrix defined in Eqs. (2) and (4)

Subscripts

j	=	jth eigenvalue
m	=	m columns or rows; parameters associate with the master degrees of freedom
p	=	system matrices of the projected model
R	=	parameters of the reduced model
s	=	parameters associate with the slave degrees of freedom

Superscripts

$i-1, i, i+1$	=	$i-1$th, ith, and $i+1$th approximation
T	=	matrix transpose
0	=	initial approximation
*	=	complex conjugate

Received 2 June 2000; revision received 11 July 2001; accepted for publication 20 July 2001.

*Professor, State Key Laboratory of Vibration, Shock and Noise; currently Research Associate, Department of Civil Engineering, University of Arkansas, 4190 Bell Engineering Center, Fayetteville, AR 72701; qu@engr.uark.edu. Member AIAA.

†Professor, Department of Civil Engineering, 4190 Bell Engineering Center; rps@engr.uark.edu.

I. Introduction

As structures to be solved for dynamic characteristics become larger or more complex, the computing time and the corresponding costs increase drastically. Hence, various techniques have been used to reduce the size of the full systems or the dimension of the structural matrices involved in the formulation. Dynamic condensation, as an efficient technique for model reduction, was first applied to large finite element models for faster computation of the natural frequencies and mode shapes. In recent years, it has been used in test-analysis model correlation, vibration control, structural dynamic optimization, dynamic modeling, and so on.[1]

Since Guyan[2] and Irons[3] first proposed this technique, many kinds of algorithms[4–15] have been developed to improve the accuracy of condensation. Among them, the iterative methods are usually more efficient than others because the dynamic condensation matrix in these approaches is updated repeatedly until the desired convergent values are obtained.[16] However, most of the dynamic condensation methods proposed in the past have been restricted to undamped models. For proportionally damped systems, the damping does not affect the eigenvectors on which the dynamic condensation matrix depends. Therefore, the dynamic condensation matrix defined for an undamped model is also valid for the corresponding proportionally damped model. Unfortunately, there are a lot of situations in which the proportional damping assumption is invalid. Examples of such cases are the structures made up of materials with different damping characteristics in different parts, structures equipped with passive and active control systems, and structures with layers of damping materials.[17]

An iterative method for dynamic condensation of viscously damped systems was proposed by Qu[1] in 1998. In this method, two governing equations for the dynamic condensation matrix, which relates the eigenvectors associated with the master and slave degrees of freedom in state space, were derived. Because the eigenvectors and eigenvalues of the reduced model are not included in the equation, it is unnecessary to solve for the eigenproblem in every iteration. Also, this method was used to the active vibration control of high buildings with active tuned mass dampers.

Most recently, a dynamic condensation approach applicable to nonclassically damped structures was proposed by Rivera et al.[17] This approach is a generalization and extension of the condensation approach in Ref. 6. In this method, the eigenproperties obtained in an iterative step are used to improve the condensation matrix in the following iterative step.

In this paper, the standard subspace iteration method for undamped models[18] is first extended to the nonclassically damped systems. A governing equation for the dynamic condensation matrix is then derived based on the extended method. Two iterative schemes are proposed to solve the governing equation. The present method has three advantages: 1) The convergence is much faster than the methods in Refs. 1 and 17, especially when the approximate values of the reduced model are close to the full model. 2) A full proof of the convergence can be made simply. 3) Because there are no parameters of the reduced model in the governing equation of the dynamic condensation matrix, it is unnecessary to calculate them during every iteration. This makes the iterative scheme much more computationally efficient, especially when the number of the master degrees of freedom is large.

II. Basic Theory of Complex Modes

The dynamic equilibrium of an n-degree-of-freedom nonclassically damped system can be expressed in a matrix form as

$$\boldsymbol{M}\ddot{\boldsymbol{X}}(t) + \boldsymbol{C}\dot{\boldsymbol{X}}(t) + \boldsymbol{K}\boldsymbol{X}(t) = \boldsymbol{F}(t) \tag{1}$$

where the mass matrix $\boldsymbol{M}$, damping matrix $\boldsymbol{C}$, and stiffness matrix $\boldsymbol{K}$ are assumed to be positive definite, positive semidefinite, and positive semidefinite, respectively. The corresponding eigenvalue problem of this system may be written in state space as

$$\boldsymbol{A}\tilde{\boldsymbol{\Psi}} = \boldsymbol{B}\tilde{\boldsymbol{\Psi}}\tilde{\boldsymbol{\Omega}} \tag{2}$$

in which the system matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are real, symmetric, and defined as

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{M} \end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} -\boldsymbol{C} & -\boldsymbol{M} \\ -\boldsymbol{M} & \boldsymbol{0} \end{bmatrix} \tag{3}$$

The complex conjugate eigenvector matrix $\tilde{\boldsymbol{\Psi}}$ and the eigenvalue or spectral matrix $\tilde{\boldsymbol{\Omega}}$ have the forms

$$\tilde{\boldsymbol{\Psi}} = \begin{bmatrix} \boldsymbol{\Psi} & \boldsymbol{\Psi}^* \\ \boldsymbol{\Psi}\boldsymbol{\Omega} & \boldsymbol{\Psi}^*\boldsymbol{\Omega}^* \end{bmatrix}, \qquad \tilde{\boldsymbol{\Omega}} = \begin{bmatrix} \boldsymbol{\Omega} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Omega}^* \end{bmatrix} \tag{4}$$

Here the eigenvalues in matrix $\tilde{\boldsymbol{\Omega}}$ are arranged in an ascending order. The eigenvector matrix is assumed to have been normalized such that

$$\tilde{\boldsymbol{\Psi}}^T\boldsymbol{A}\tilde{\boldsymbol{\Psi}} = \tilde{\boldsymbol{\Omega}}, \qquad \tilde{\boldsymbol{\Psi}}^T\boldsymbol{B}\tilde{\boldsymbol{\Psi}} = \boldsymbol{I} \tag{5}$$

If only the lower m eigenpairs are considered in Eq. (2), one has

$$\boldsymbol{A}\tilde{\boldsymbol{\Psi}}_m = \boldsymbol{B}\tilde{\boldsymbol{\Psi}}_m\tilde{\boldsymbol{\Omega}}_{mm} \tag{6}$$

or in an expanded form

$$\begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{M} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Psi}_m & \boldsymbol{\Psi}_m^* \\ \boldsymbol{\Psi}_m\boldsymbol{\Omega}_{mm} & \boldsymbol{\Psi}_m^*\boldsymbol{\Omega}_{mm}^* \end{bmatrix} = \begin{bmatrix} -\boldsymbol{C} & -\boldsymbol{M} \\ -\boldsymbol{M} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Psi}_m & \boldsymbol{\Psi}_m^* \\ \boldsymbol{\Psi}_m\boldsymbol{\Omega}_{mm} & \boldsymbol{\Psi}_m^*\boldsymbol{\Omega}_{mm}^* \end{bmatrix} \begin{bmatrix} \boldsymbol{\Omega}_{mm} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Omega}_{mm}^* \end{bmatrix} \tag{7}$$

in which the dimensions of submatrices $\boldsymbol{\Psi}_m$ and $\boldsymbol{\Omega}_{mm}$ are $n \times m$ and $m \times m$, respectively.

III. Iterative Methods of Qu[1] and Rivera[17]

In the dynamic condensation technique, the total degrees of freedom n of the full model are usually divided into the master degrees of freedom m, which will be retained in the reduced model, and the slave degrees of freedom s, which will be omitted. Based on this division, Eq. (6) can be rewritten in a partitioned form as

$$\begin{bmatrix} \boldsymbol{A}_{mm} & \boldsymbol{A}_{ms} \\ \boldsymbol{A}_{sm} & \boldsymbol{A}_{ss} \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{\Psi}}_{mm} \\ \tilde{\boldsymbol{\Psi}}_{sm} \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}_{mm} & \boldsymbol{B}_{ms} \\ \boldsymbol{B}_{sm} & \boldsymbol{B}_{ss} \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{\Psi}}_{mm} \\ \tilde{\boldsymbol{\Psi}}_{sm} \end{bmatrix} \tilde{\boldsymbol{\Omega}}_{mm} \tag{8}$$

where the submatrices are given by

$$\boldsymbol{A}_{mm} = \begin{bmatrix} \boldsymbol{K}_{mm} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{M}_{mm} \end{bmatrix}, \qquad \boldsymbol{A}_{ms} = \boldsymbol{A}_{sm}^T = \begin{bmatrix} \boldsymbol{K}_{ms} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{M}_{ms} \end{bmatrix}$$

$$\boldsymbol{A}_{ss} = \begin{bmatrix} \boldsymbol{K}_{ss} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{M}_{ss} \end{bmatrix}, \qquad \boldsymbol{B}_{mm} = \begin{bmatrix} -\boldsymbol{C}_{mm} & -\boldsymbol{M}_{mm} \\ -\boldsymbol{M}_{mm} & \boldsymbol{0} \end{bmatrix}$$

$$\boldsymbol{B}_{ms} = \boldsymbol{B}_{sm}^T = \begin{bmatrix} -\boldsymbol{C}_{ms} & -\boldsymbol{M}_{ms} \\ -\boldsymbol{M}_{ms} & \boldsymbol{0} \end{bmatrix}, \qquad \boldsymbol{B}_{ss} = \begin{bmatrix} -\boldsymbol{C}_{ss} & -\boldsymbol{M}_{ss} \\ -\boldsymbol{M}_{ss} & \boldsymbol{0} \end{bmatrix}$$

$$\tilde{\boldsymbol{\Psi}}_{mm} = \begin{bmatrix} \boldsymbol{\Psi}_{mm} & \boldsymbol{\Psi}_{mm}^* \\ \boldsymbol{\Psi}_{mm}\boldsymbol{\Omega}_{mm} & \boldsymbol{\Psi}_{mm}^*\boldsymbol{\Omega}_{mm}^* \end{bmatrix}, \qquad \tilde{\boldsymbol{\Psi}}_{sm} = \begin{bmatrix} \boldsymbol{\Psi}_{sm} & \boldsymbol{\Psi}_{sm}^* \\ \boldsymbol{\Psi}_{sm}\boldsymbol{\Omega}_{mm} & \boldsymbol{\Psi}_{sm}^*\boldsymbol{\Omega}_{mm}^* \end{bmatrix}$$

$$\tilde{\boldsymbol{\Omega}}_{mm} = \begin{bmatrix} \boldsymbol{\Omega}_{mm} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Omega}_{mm}^* \end{bmatrix} \tag{9}$$

Expanding the lower part of Eq. (8) and rearranging the result yields

$$\tilde{\boldsymbol{\Psi}}_{sm} = \boldsymbol{A}_{ss}^{-1}(\boldsymbol{B}_{sm}\tilde{\boldsymbol{\Psi}}_{mm}\tilde{\boldsymbol{\Omega}}_{mm} + \boldsymbol{B}_{ss}\tilde{\boldsymbol{\Psi}}_{sm}\tilde{\boldsymbol{\Omega}}_{mm} - \boldsymbol{A}_{sm}\tilde{\boldsymbol{\Psi}}_{mm}) \tag{10}$$

According to the definition of the dynamic condensation matrix of nonclassically damped systems,[1,17] that is,

$$\tilde{\boldsymbol{\Psi}}_{sm} = \boldsymbol{R}\tilde{\boldsymbol{\Psi}}_{mm} \tag{11}$$

the governing equation of the dynamic condensation matrix $\boldsymbol{R}$ can be obtained from Eq. (10) as

$$\boldsymbol{R} = \boldsymbol{A}_{ss}^{-1}\left[(\boldsymbol{B}_{sm} + \boldsymbol{B}_{ss}\boldsymbol{R})\tilde{\boldsymbol{\Psi}}_{mm}\tilde{\boldsymbol{\Omega}}_{mm}\tilde{\boldsymbol{\Psi}}_{mm}^{-1} - \boldsymbol{A}_{sm}\right] \tag{12}$$

To make the computation efficient, the following two equations are usually used instead of Eq. (12):

$$\boldsymbol{R} = \boldsymbol{A}_{ss}^{-1}\left[(\boldsymbol{B}_{sm} + \boldsymbol{B}_{ss}\boldsymbol{R})\tilde{\boldsymbol{\Psi}}_{mm}\tilde{\boldsymbol{\Psi}}_{mm}^T\boldsymbol{A}_R - \boldsymbol{A}_{sm}\right] \tag{13}$$

$$\boldsymbol{R} = \boldsymbol{A}_{ss}^{-1}\left[(\boldsymbol{B}_{sm} + \boldsymbol{B}_{ss}\boldsymbol{R})\boldsymbol{B}_R^{-1}\boldsymbol{A}_R - \boldsymbol{A}_{sm}\right] \tag{14}$$

where the reduced system matrices $\boldsymbol{A}_R$ and $\boldsymbol{B}_R$ are defined as

$$\boldsymbol{A}_R = \boldsymbol{A}_{mm} + \boldsymbol{R}^T\boldsymbol{A}_{sm} + \boldsymbol{A}_{ms}\boldsymbol{R} + \boldsymbol{R}^T\boldsymbol{A}_{ss}\boldsymbol{R} \tag{15a}$$

$$\boldsymbol{B}_R = \boldsymbol{B}_{mm} + \boldsymbol{R}^T\boldsymbol{B}_{sm} + \boldsymbol{B}_{ms}\boldsymbol{R} + \boldsymbol{R}^T\boldsymbol{B}_{ss}\boldsymbol{R} \tag{15b}$$

Because the governing equations (13) and (14) are nonlinear, it is difficult to solve them directly. The iterative forms of these two equations for $i = 1, 2, \ldots$, are given by[1]

$$\boldsymbol{R}^{(i)} = \boldsymbol{A}_{ss}^{-1}\left[\left(\boldsymbol{B}_{sm} + \boldsymbol{B}_{ss}\boldsymbol{R}^{(i-1)}\right)\tilde{\boldsymbol{\Psi}}_{mm}^{(i-1)}\left(\tilde{\boldsymbol{\Psi}}_{mm}^{(i-1)}\right)^{T}\boldsymbol{A}_{R}^{(i-1)} - \boldsymbol{A}_{sm}\right] \tag{16}$$

$$\boldsymbol{R}^{(i)} = \boldsymbol{A}_{ss}^{-1}\left[\left(\boldsymbol{B}_{sm} + \boldsymbol{B}_{ss}\boldsymbol{R}^{(i-1)}\right)\left(\boldsymbol{B}_{R}^{(i-1)}\right)^{-1}\boldsymbol{A}_{R}^{(i-1)} - \boldsymbol{A}_{sm}\right] \tag{17}$$

The initial approximation of the dynamic condensation matrix $\boldsymbol{R}^{(0)}$ is given by[1]

$$\boldsymbol{R}^{(0)} = -\boldsymbol{A}_{ss}^{-1}\boldsymbol{A}_{sm} = \begin{bmatrix} -\boldsymbol{K}_{ss}^{-1}\boldsymbol{K}_{sm} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{M}_{ss}^{-1}\boldsymbol{M}_{sm} \end{bmatrix} \tag{18}$$

An alternative initial approximation, that is,

$$\boldsymbol{R}^{(0)} = \begin{bmatrix} -\boldsymbol{K}_{ss}^{-1}\boldsymbol{K}_{sm} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{K}_{ss}^{-1}\boldsymbol{K}_{sm} \end{bmatrix} \tag{19}$$

was used by Rivera et al.[17] It has the advantage that the calculation of the inverse of matrix $\boldsymbol{M}_{ss}$ is avoided.

IV. Present Method

A. Subspace Iteration Method for Complex Eigenproblems

The subspace iteration method is widely used for the computation of a few smallest eigenvalues and eigenvectors of large eigenproblems. The standard subspace iteration method, developed by Bathe and Wilson,[18] is a direct iterative method for symmetric matrices. It combines a simultaneous inverse iteration and a Rayleigh–Ritz procedure. This standard method is extended to evaluate the eigenpairs of nonclassically damped systems in what follows.

Choose a set of linearly independent $2n$-dimensional vectors, and construct a subspace $\tilde{\boldsymbol{\Psi}}_{m}^{(0)}$ in which the columns are occupied by the vectors. This subspace is usually considered as an initial approximation of the eigenvectors. For $i = 1, 2, \ldots$, the following two steps are applied to solve for the $i+1$th approximation of eigenvalues and eigenvectors.

1) A set of new subspace $\boldsymbol{X}_{m}^{(i+1)}$ is obtained by simultaneous inverse iteration, that is,

$$\boldsymbol{A}\boldsymbol{X}_{m}^{(i+1)} = \boldsymbol{B}\tilde{\boldsymbol{\Psi}}_{m}^{(i)} \tag{20}$$

If the iterations proceeded using $\boldsymbol{X}_{m}^{(i+1)}$ as the next estimation of the subspace, the subspace would collapse to a subspace of dimension one and only contains the eigenvector corresponding to the lowest eigenvalue. Hence, the Rayleigh–Ritz procedure is adopted.

2) Compute the projections of matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ in the subspace spanned by $\boldsymbol{X}_{m}^{(i+1)}$:

$$\boldsymbol{A}_{P}^{(i+1)} = \left(\boldsymbol{X}_{m}^{(i+1)}\right)^{T}\boldsymbol{A}\boldsymbol{X}_{m}^{(i+1)}, \qquad \boldsymbol{B}_{P}^{(i+1)} = \left(\boldsymbol{X}_{m}^{(i+1)}\right)^{T}\boldsymbol{B}\boldsymbol{X}_{m}^{(i+1)} \tag{21}$$

Then solve for the projected eigenproblem given by

$$\boldsymbol{A}_{P}^{(i+1)}\tilde{\boldsymbol{Q}}^{(i+1)} = \boldsymbol{B}_{P}^{(i+1)}\tilde{\boldsymbol{Q}}^{(i+1)}\tilde{\Lambda}^{(i+1)} \tag{22}$$

where $\tilde{\boldsymbol{Q}}^{(i+1)}$ and $\tilde{\Lambda}^{(i+1)}$ are the $(i+1)$th approximate eigenvector and eigenvalue matrices of the projected model. Finally, the $(i+1)$th approximate eigenvector matrix is given by

$$\tilde{\boldsymbol{\Psi}}_{m}^{(i+1)} = \boldsymbol{X}_{m}^{(i+1)}\tilde{\boldsymbol{Q}}^{(i+1)} \tag{23}$$

Eigenvector matrix $\tilde{\boldsymbol{\Psi}}_{m}^{(i+1)}$ is used to calculate the next approximate eigenvalues and eigenvectors until they converge, that is,

$$\frac{\left|\lambda_{j}^{(i+1)} - \lambda_{j}^{(i)}\right|}{\left|\lambda_{j}^{(i+1)}\right|} \le \varepsilon, \qquad j = 1, 2, \ldots, p \le m \tag{24}$$

If the first p eigenvalues converge, exit the loop.

B. Governing Equation for Dynamic Condensation Matrix

If the total degrees of freedom of a model are divided into the master and slave degrees of freedom as mentioned earlier, Eq. (23) can be rewritten in a partitioned form as

$$\begin{bmatrix} \tilde{\boldsymbol{\Psi}}_{mm}^{(i+1)} \\ \tilde{\boldsymbol{\Psi}}_{sm}^{(i+1)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}_{mm}^{(i+1)}\tilde{\boldsymbol{Q}}^{(i+1)} \\ \boldsymbol{X}_{sm}^{(i+1)}\tilde{\boldsymbol{Q}}^{(i+1)} \end{bmatrix} \tag{25}$$

where submatrices $\tilde{\boldsymbol{\Psi}}_{mm}^{(i+1)}$ $(2m \times 2m)$ and $\tilde{\boldsymbol{\Psi}}_{sm}^{(i+1)}$ $(2s \times 2m)$ are defined as

$$\tilde{\boldsymbol{\Psi}}_{mm}^{(i+1)} = \begin{bmatrix} \boldsymbol{\Psi}_{mm}^{(i+1)} & \left(\boldsymbol{\Psi}_{mm}^{(i+1)}\right)^{*} \\ \boldsymbol{\Psi}_{mm}^{(i+1)}\boldsymbol{\Omega}_{mm}^{(i+1)} & \left(\boldsymbol{\Psi}_{mm}^{(i+1)}\right)^{*}\left(\boldsymbol{\Omega}_{mm}^{(i+1)}\right)^{*} \end{bmatrix}$$

$$\tilde{\boldsymbol{\Psi}}_{sm}^{(i+1)} = \begin{bmatrix} \boldsymbol{\Psi}_{sm}^{(i+1)} & \left(\boldsymbol{\Psi}_{sm}^{(i+1)}\right)^{*} \\ \boldsymbol{\Psi}_{sm}^{(i+1)}\boldsymbol{\Omega}_{mm}^{(i+1)} & \left(\boldsymbol{\Psi}_{sm}^{(i+1)}\right)^{*}\left(\boldsymbol{\Omega}_{mm}^{(i+1)}\right)^{*} \end{bmatrix} \tag{26}$$

According to the definition of the dynamic condensation matrix in Eq. (11), one has

$$\boldsymbol{R}^{(i+1)} = \tilde{\boldsymbol{\Psi}}_{sm}^{(i+1)}\left(\tilde{\boldsymbol{\Psi}}_{mm}^{(i+1)}\right)^{-1} \tag{27}$$

Introducing Eq. (25) into Eq. (27) yields

$$\boldsymbol{R}^{(i+1)} = \boldsymbol{X}_{sm}^{(i+1)}\tilde{\boldsymbol{Q}}^{(i+1)}\left(\boldsymbol{X}_{mm}^{(i+1)}\tilde{\boldsymbol{Q}}^{(i+1)}\right)^{-1} \tag{28}$$

which can be expressed in concise form as

$$\boldsymbol{R}^{(i+1)} = \boldsymbol{X}_{sm}^{(i+1)}\left(\boldsymbol{X}_{mm}^{(i+1)}\right)^{-1} \tag{29}$$

It is shown clearly in Eq. (29) that the Rayleigh–Ritz procedure does not affect the dynamic condensation matrix.

Equation (20) can be rewritten as

$$\boldsymbol{X}_{m}^{(i+1)} = \boldsymbol{G}\tilde{\boldsymbol{\Psi}}_{m}^{(i)} \tag{30}$$

in which matrix $\boldsymbol{G}$ is defined as

$$\boldsymbol{G} = \boldsymbol{A}^{-1}\boldsymbol{B} \tag{31}$$

By considering Eq. (3), one has

$$\boldsymbol{G} = \begin{bmatrix} \boldsymbol{E} & \boldsymbol{F} \\ \boldsymbol{I} & \boldsymbol{0} \end{bmatrix} \tag{32}$$

$$\boldsymbol{E} = -\boldsymbol{K}^{-1}\boldsymbol{C}, \qquad \boldsymbol{F} = -\boldsymbol{K}^{-1}\boldsymbol{M} \tag{33}$$

$\boldsymbol{I}$ and $\boldsymbol{0}$ $(n \times n)$ are a unity and a zero matrix, respectively. Based on the division of the degrees of freedom, Eq. (30) may be expressed in partitioned form as

$$\begin{bmatrix} \boldsymbol{X}_{mm}^{(i+1)} \\ \boldsymbol{X}_{sm}^{(i+1)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{G}_{11} & \boldsymbol{G}_{12} \\ \boldsymbol{G}_{21} & \boldsymbol{G}_{22} \end{bmatrix}\begin{bmatrix} \tilde{\boldsymbol{\Psi}}_{mm}^{(i)} \\ \tilde{\boldsymbol{\Psi}}_{sm}^{(i)} \end{bmatrix} \tag{34}$$

where the submatrices $\boldsymbol{G}_{11}(2m \times 2m)$, $\boldsymbol{G}_{12}(2m \times 2s)$, $\boldsymbol{G}_{21}(2s \times 2m)$, and $\boldsymbol{G}_{22}(2s \times 2s)$ are defined as

$$\boldsymbol{G}_{11} = \begin{bmatrix} \boldsymbol{E}_{mm} & \boldsymbol{F}_{mm} \\ \boldsymbol{I}_{mm} & \boldsymbol{0} \end{bmatrix}, \qquad \boldsymbol{G}_{12} = \boldsymbol{G}_{21}^{T} = \begin{bmatrix} \boldsymbol{E}_{ms} & \boldsymbol{F}_{ms} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}$$

$$\boldsymbol{G}_{22} = \begin{bmatrix} \boldsymbol{E}_{ss} & \boldsymbol{F}_{ss} \\ \boldsymbol{I}_{ss} & \boldsymbol{0} \end{bmatrix} \tag{35}$$

Using the definition of dynamic condensation matrix in Eq. (11), one has

$$\begin{bmatrix} \tilde{\boldsymbol{\Psi}}_{mm}^{(i)} \\ \tilde{\boldsymbol{\Psi}}_{sm}^{(i)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{R}^{(i)} \end{bmatrix}\tilde{\boldsymbol{\Psi}}_{mm}^{(i)} \tag{36}$$

By introducing Eq. (36) into Eq. (34), we have

$$\begin{bmatrix} \boldsymbol{X}_{mm}^{(i+1)} \\ \boldsymbol{X}_{sm}^{(i+1)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{G}_{11} & \boldsymbol{G}_{12} \\ \boldsymbol{G}_{21} & \boldsymbol{G}_{22} \end{bmatrix}\begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{R}^{(i)} \end{bmatrix}\tilde{\boldsymbol{\Psi}}_{mm}^{(i)} \tag{37}$$

Equation (37) is equivalent to the following two equations:

$$\boldsymbol{X}_{mm}^{(i+1)} = \left(\boldsymbol{G}_{11} + \boldsymbol{G}_{12}\boldsymbol{R}^{(i)}\right)\tilde{\boldsymbol{\Psi}}_{mm}^{(i)} \tag{38a}$$

$$\boldsymbol{X}_{sm}^{(i+1)} = \left(\boldsymbol{G}_{21} + \boldsymbol{G}_{22}\boldsymbol{R}^{(i)}\right)\tilde{\boldsymbol{\Psi}}_{mm}^{(i)} \tag{38b}$$

Substituting Eq. (38) into Eq. (29) results in

$$\boldsymbol{R}^{(i+1)} = \left(\boldsymbol{G}_{21} + \boldsymbol{G}_{22}\boldsymbol{R}^{(i)}\right)\left(\boldsymbol{G}_{11} + \boldsymbol{G}_{12}\boldsymbol{R}^{(i)}\right)^{-1} \tag{39}$$

If we let $i = -1$ and $\boldsymbol{R}^{(-1)} = \boldsymbol{0}$ in Eq. (39), we have

$$\boldsymbol{R}^{(0)} = \boldsymbol{G}_{21}\boldsymbol{G}_{11}^{-1} \tag{40}$$

Equations (39) and (40) are the governing equations of the dynamic condensation matrix. It can be seen clearly from them that the dynamic condensation matrix has nothing to do with the system matrices and eigenpairs of the reduced model. Therefore, it is unnecessary to calculate them during iteration, which makes the present method very computationally efficient.

C. Iterative Schemes for the Dynamic Condensation Matrix

Classical Iterative Scheme

The iterative procedure used in Refs. 1 and 17 is implemented to solve for the dynamic condensation matrix governed by Eq. (39). The main steps are as follows:

1) Choose the master degrees of freedom, and compute all of the submatrices to be used in the following.

2) Calculate the initial approximation of the dynamic condensation $\boldsymbol{R}^{(0)}$ by using Eq. (40).

3) For $i = 0, 1, 2, 3, \ldots$, begin the iteration:

a) Calculate the $(i+1)$th approximate dynamic condensation matrix $\boldsymbol{R}^{(i+1)}$ using Eq. (39).

b) Construct the system matrices of the reduced model using Eq. (15).

c) Solve for the eigenproblem of the reduced model:

$$\boldsymbol{A}_R^{(i+1)}\tilde{\boldsymbol{\Psi}}_{mm}^{(i+1)} = \boldsymbol{B}_R^{(i+1)}\tilde{\boldsymbol{\Psi}}_{mm}^{(i+1)}\tilde{\boldsymbol{\Omega}}_{mm}^{(i+1)} \tag{41}$$

d) Check the convergence for the real and imaginary parts of the eigenvalues by using the following convergent criterion:

$$\frac{\left|\alpha_j^{(i+1)} - \alpha_j^{(i)}\right|}{\left|\alpha_j^{(i+1)}\right|} \le \varepsilon_1, \qquad j = 1, 2, \ldots, p \le m \tag{42}$$

where α denotes the real and imaginary parts of the complex eigenvalues, respectively. If the first p eigenvalues converge, exit the loop.

4) Output the dynamic condensation matrix $\boldsymbol{R}^{(i+1)}$ and system matrices $\boldsymbol{A}_R^{(i+1)}$ and $\boldsymbol{B}_R^{(i+1)}$ of the reduced model.

In this iterative scheme, the eigenvalues as well as the eigenvectors are computed during iteration. Actually, this is just one application of the dynamic condensation technique. What is much more important in vibration engineering is that a reduced model ($\boldsymbol{A}_R$ and $\boldsymbol{B}_R$) is defined by this technique. The reduced model has the following two special characteristics that make it very useful in dynamic analysis such as test-analysis model correlation, active vibration, etc. 1) The eigenvalues and eigenvectors of the reduced model are very close to those that result from the full model. Hence, the reduced model can represent the full model in that frequency range. 2) The reduced model is defined in the subspace of the original space used by the full model. This means each coordinate of the subspace has its physical meaning.

Clearly, there are three types of major computational work within each of the iterations. They are to evaluate the $(i+1)$th approximate dynamic condensation matrix $\boldsymbol{R}^{(i+1)}$, to construct the $(i+1)$th approximate system matrices $\boldsymbol{A}_R^{(i+1)}$ and $\boldsymbol{B}_R^{(i+1)}$ of the reduced model, and to solve for the eigenproblem of the reduced model. Assume the computational work for these three types are $W1$, $W2$, and $W3$, respectively. The total work for one iteration is $W1 + W2 + W3$ and $k(W1 + W2 + W3)$ for k iterations.

If Eq. (40) in step 2 is replaced by Eqs. (18) and (19) and Eq. (39) in step 3a is replaced by Eqs. (17) and (16), the iterative schemes used by Qu[1] and Rivera[17] result, respectively. Therefore, the computational work used in the iterative schemes[1,17] is very close to that used in the preceding scheme.

As shown in Eq. (31), because the matrix $\boldsymbol{G}$ is defined by the system matrices of the full model directly, the dynamic condensation matrix is only dependent on itself, as shown in Eq. (39). This means the system matrices as well as the eigenpairs of the reduced model have no effect on the iteration. We do not have to compute them within every iteration. Therefore, the following iterative scheme is presented.

Iterative Scheme 1

1) Choose the master degrees of freedom and compute all the submatrices to be used in the following.

2) Calculate the initial approximation of the dynamic condensation $\boldsymbol{R}^{(0)}$ by using Eq. (40).

3) For $i = 0, k, 2k, 3k, \ldots, (k > 1)$, begin the iteration:

a) Calculate the $(i+k)$th approximate dynamic condensation matrix $\boldsymbol{R}^{(i+k)}$ by iterating Eq. (39) for k times.

b) Calculate the system matrices of the reduced model using Eq. (15).

c) Solve for the eigenproblem of the reduced model

$$\boldsymbol{A}_R^{(i+k)}\tilde{\boldsymbol{\Psi}}_{mm}^{(i+k)} = \boldsymbol{B}_R^{(i+k)}\tilde{\boldsymbol{\Psi}}_{mm}^{(i+k)}\tilde{\boldsymbol{\Omega}}_{mm}^{(i+k)} \tag{43}$$

d) Check the convergence for the real and imaginary parts of the eigenvalues using the convergent criterion

$$\frac{\left|\alpha_j^{(i+k)} - \alpha_j^{(i)}\right|}{\left|\alpha_j^{(i+k)}\right|} \le \varepsilon_1, \qquad j = 1, 2, \ldots, p \le m \tag{44}$$

If the first p eigenvalues converge, exit the loop.

4) Output the dynamic condensation matrix $\boldsymbol{R}^{(i+k)}$ and system matrices $\boldsymbol{A}_R^{(i+k)}$ and $\boldsymbol{B}_R^{(i+k)}$ of the reduced model.

Clearly, iterative scheme 1 becomes the classical iterative scheme if $k = 1$ in the former. Because the computation of the system matrices of the reduced model and the corresponding eigenproblem does not affect the dynamic condensation matrix, the dynamic condensation matrix $\boldsymbol{R}^{(i+k)}$ and the system matrices $\boldsymbol{A}_R^{(i+k)}$ and $\boldsymbol{B}_R^{(i+k)}$ of the reduced model resulting from the two schemes should be identical.

As mentioned earlier, the major computational work for obtaining the reduced model ($\boldsymbol{A}_R^{(i+k)}$ and $\boldsymbol{B}_R^{(i+k)}$) is $k(W1 + W2 + W3)$. It is $kW1 + W2 + W3$ if iterative scheme 1 is used. Clearly, $(k-1)(W2 + W3)$ computational work may be saved for k iterations and $(k-1)(W2 + W3)/k$ for one iteration. Here, $W3$ is the computation of the eigenproblem of the reduced model in state space. As we know, the computation of the eigenpairs is usually very expensive, especially when the size of the reduced model or the number of the master degrees of freedom is large.[19] Therefore, the computational effort required in iterative scheme 1 is much less than the classical iterative scheme and the previous schemes[1,17] if $k > 1$ and the size of the reduced model is big.

In iterative scheme 1, the system matrices of the reduced model and the corresponding eigenpairs are still to be computed after a couple of iterations. Hence, another iterative scheme, iterative scheme 2, is presented.

Iterative Scheme 2

Steps 1 and 2 are similar to scheme 1.

3) For $i = 0, 1, 2, 3, \ldots$, begin the iteration:

a) Calculate the $(i+1)$th approximate dynamic condensation matrix $\boldsymbol{R}^{(i+k)}$ by using Eq. (39).

b) Check the convergence by using criterion

$$\text{error} = 1 - \frac{\left(\boldsymbol{r}_j^{(i+1)}\right)^T \cdot \boldsymbol{r}_j^{(i)}}{\left\|\boldsymbol{r}_j^{(i+1)}\right\|_2 \cdot \left\|\boldsymbol{r}_j^{(i)}\right\|_2} \le \varepsilon_2, \qquad j = 1, 2, \ldots, m \tag{45}$$

where $\boldsymbol{r}_j^{(i)}$ and $\boldsymbol{r}_j^{(i+1)}$ are the jth column vectors of the ith and $(i+1)$th approximate dynamic condensation matrix, respectively. If the m column vectors converge, exit the loop.

4) Calculate the system matrices of the reduced model.
5) Solve for the eigenproblem of the reduced model if necessary.
6) Output the dynamic condensation matrix and system matrices.

In this scheme, the system matrices of the reduced model are only to be calculated after the dynamic condensation matrix converges. The eigenpairs are to be computed only when it is necessary. Therefore, this scheme is a little more computationally efficient than scheme 1.

D. Discussion on the Convergence

Scheme 2 is reproduced in a form that would be convenient for the discussion of the convergence.

1) Suppose the dynamic condensation matrix is a zero matrix, and construct subspace $\boldsymbol{X}_m$ as

$$\boldsymbol{X}_m = \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{R} \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{0} \end{bmatrix} \tag{46a}$$

2) Calculate the initially approximate dynamic condensation matrix by using the following two equations:

$$\boldsymbol{X}_m^{(0)} = \begin{bmatrix} \boldsymbol{X}_{mm}^{(0)} \\ \boldsymbol{X}_{sm}^{(0)} \end{bmatrix} = \boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{X}_m \tag{46b}$$

$$\boldsymbol{R}^{(0)} = \boldsymbol{X}_{sm}^{(0)}\left(\boldsymbol{X}_{mm}^{(0)}\right)^{-1} \tag{46c}$$

3) For $i = 0, 1, 2, \ldots$, begin the iteration. According to Eq. (29), the $(i+1)$th approximate condensation matrix can be obtained from the following two equations:

$$\boldsymbol{X}_m^{(i+1)} = \boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{X}_m^{(i)} \tag{46d}$$

$$\boldsymbol{R}^{(i+1)} = \boldsymbol{X}_{sm}^{(i+1)}\left(\boldsymbol{X}_{mm}^{(i+1)}\right)^{-1} \tag{46e}$$

The following steps are similar to those in scheme 2. Clearly, Eqs. (46c) and (46e) are equivalent to Eqs. (40) and (39).

Suppose the subspace $\boldsymbol{X}_m$ can be expressed as

$$\boldsymbol{X}_m = \tilde{\boldsymbol{\Psi}}\boldsymbol{D} \tag{47}$$

where $\boldsymbol{D}$ is a coefficient matrix of $2n \times 2m$. Introducing Eq. (47) into Eq. (46b) results in

$$\boldsymbol{X}_m^{(0)} = \boldsymbol{A}^{-1}\boldsymbol{B}\tilde{\boldsymbol{\Psi}}\boldsymbol{D} \tag{48}$$

When the orthogonalities of the eigenvector matrix (5) are considered, Eq. (48) can be written as

$$\boldsymbol{X}_m^{(0)} = \tilde{\boldsymbol{\Psi}}\tilde{\boldsymbol{\Omega}}^{-1}\boldsymbol{D} \tag{49}$$

Equation (49) can be partitioned as

$$\begin{bmatrix} \boldsymbol{X}_{mm}^{(0)} \\ \boldsymbol{X}_{sm}^{(0)} \end{bmatrix} = \begin{bmatrix} \tilde{\boldsymbol{\Psi}}_{mm} & \tilde{\boldsymbol{\Psi}}_{ms} \\ \tilde{\boldsymbol{\Psi}}_{sm} & \tilde{\boldsymbol{\Psi}}_{ss} \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{\Omega}}_{mm}^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \tilde{\boldsymbol{\Omega}}_{ss}^{-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{D}_{mm} \\ \boldsymbol{D}_{sm} \end{bmatrix} \tag{50}$$

which is equivalent to the following two equations:

$$\boldsymbol{X}_{mm}^{(0)} = \tilde{\boldsymbol{\Psi}}_{mm}\tilde{\boldsymbol{\Omega}}_{mm}^{-1}\boldsymbol{D}_{mm} + \tilde{\boldsymbol{\Psi}}_{ms}\tilde{\boldsymbol{\Omega}}_{ss}^{-1}\boldsymbol{D}_{sm} \tag{51a}$$

$$\boldsymbol{X}_{sm}^{(0)} = \tilde{\boldsymbol{\Psi}}_{sm}\tilde{\boldsymbol{\Omega}}_{mm}^{-1}\boldsymbol{D}_{mm} + \tilde{\boldsymbol{\Psi}}_{ss}\tilde{\boldsymbol{\Omega}}_{ss}^{-1}\boldsymbol{D}_{sm} \tag{51b}$$

Introducing Eq. (51) into Eq. (46c) leads to

$$\boldsymbol{R}^{(0)} = \left(\tilde{\boldsymbol{\Psi}}_{sm}\tilde{\boldsymbol{\Omega}}_{mm}^{-1}\boldsymbol{D}_{mm} + \tilde{\boldsymbol{\Psi}}_{ss}\tilde{\boldsymbol{\Omega}}_{ss}^{-1}\boldsymbol{D}_{sm}\right) \times \left(\tilde{\boldsymbol{\Psi}}_{mm}\tilde{\boldsymbol{\Omega}}_{mm}^{-1}\boldsymbol{D}_{mm} + \tilde{\boldsymbol{\Psi}}_{ms}\tilde{\boldsymbol{\Omega}}_{ss}^{-1}\boldsymbol{D}_{sm}\right) \tag{52}$$

Based on the same derivative procedure, the ith approximation of the dynamic condensation matrix is

$$\boldsymbol{R}^{(i)} = \left(\tilde{\boldsymbol{\Psi}}_{sm}\tilde{\boldsymbol{\Omega}}_{mm}^{-i-1}\boldsymbol{D}_{mm} + \tilde{\boldsymbol{\Psi}}_{ss}\tilde{\boldsymbol{\Omega}}_{ss}^{-i-1}\boldsymbol{D}_{sm}\right) \times \left(\tilde{\boldsymbol{\Psi}}_{mm}\tilde{\boldsymbol{\Omega}}_{mm}^{-i-1}\boldsymbol{D}_{mm} + \tilde{\boldsymbol{\Psi}}_{ms}\tilde{\boldsymbol{\Omega}}_{ss}^{-i-1}\boldsymbol{D}_{sm}\right)^{-1} \tag{53}$$

Because the moduli of all of the diagonal elements in matrix $\tilde{\boldsymbol{\Omega}}_{ss}$ are greater than those in matrix $\tilde{\boldsymbol{\Omega}}_{mm}$, one has

$$\boldsymbol{R}^{(i)} \to \left(\tilde{\boldsymbol{\Psi}}_{sm}\tilde{\boldsymbol{\Omega}}_{mm}^{-i-1}\boldsymbol{D}_{mm}\right)\left(\tilde{\boldsymbol{\Psi}}_{mm}\tilde{\boldsymbol{\Omega}}_{mm}^{-i-1}\boldsymbol{D}_{mm}\right)^{-1} \qquad (i \to \infty) \tag{54}$$

that is,

$$\boldsymbol{R}^{(i)} \to \tilde{\boldsymbol{\Psi}}_{sm}\tilde{\boldsymbol{\Psi}}_{mm}^{-1} \qquad (i \to \infty) \tag{55}$$

in which $\tilde{\boldsymbol{\Psi}}_{sm}\tilde{\boldsymbol{\Psi}}_{mm}^{-1}$ is the exact value of the dynamic condensation matrix.

V. Numerical Examples

Two factors affect the efficiency of an iterative method. One is the computational effort at each iteration, and the other is the convergence rate of each iteration. As discussed in the preceding section, $(k-1)(W2+W3)/k$ computational work may be saved for each iteration if iterative scheme 1 rather than the classical iterative scheme or previous schemes[1,17] is used. The computational work of the iterative scheme 2 is a little less than scheme 1. For the first factor, the proposed method or scheme is, therefore, superior to the previous approaches.[1,17] The remainder is the second factor. Because it is very difficult to discuss this factor theoretically, two numerical examples are included. We will compare the results of the reduced models obtained from difference condensation approaches iteration by iteration. Therefore, only the classical iterative scheme will be applied.

A. Mass–Damper–Stiffness System

A discrete mass–damper–spring system, shown in Fig. 1, is considered. In this system, $m_i = 1.0$ kg, $c_i = 0.5i$ N·s/m, $k_i = 200i$ N/m, $i = 1, 2, \ldots, 20$. It has a total of 20 degrees of freedom. The lower four complex eigenvalues are $-0.017635 \pm j3.75602$, $-0.093032 \pm j8.62651$, $-0.229146 \pm j13.5375$, and $-0.426827 \pm j18.4737$. The 1st, 6th, 11th, and 16th degrees of freedom are selected as the master degrees of freedom when condensed. The errors of the eigenvalues of the reduced model in the former 10 iterations are listed in Table 1. The error is defined as

$$\text{error} = \frac{\alpha_j^{(i)} - \alpha_j^e}{\alpha_j^e}, \qquad j = 1, 2, \ldots, m \tag{56}$$

in which $\alpha_j^{(i)}$ and α_j^e are the real/imaginary parts of the ith approximate and exact eigenvalues, respectively. The subscript denotes the jth eigenvalue. Here, the eigenvalues are just used to demonstrate how the reduced model closes to the full model and are not used for eigenvalues themselves.

As shown clearly from the results in Table 1, the errors of the initial approximations are very large, especially for the real parts of the eigenvalues. All of the errors of the real parts are greater than 100%. Clearly, the corresponding eigenvalues or the reduced model is meaningless. The real parts of the eigenvalues resulting from the reducing model converge to the exact result quickly when iteration is applied. After 10 iterations, the errors reduce to less than 1000th of the initial approximations. The imaginary parts of the eigenvalues converge to the exact result consistently during iterating. These errors are all larger than zero, which means the frequencies of the reduced model are larger than the exact result, and the reduced model closes to the full model from above. After 10 iterations, the reduced model A_R and B_R, which is described by the 1st, 6th, 11th, and 16th degrees of freedom, can accurately represent the full model in low-frequency range with the highest error 0.5%. Therefore, it can be used directly in the test-analysis model correlation, active vibration control, and so on.

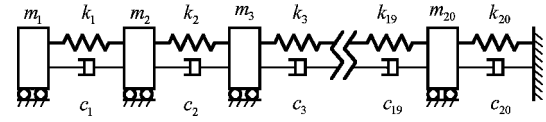


Fig. 1 Schematic of a mass–damping–spring system.

Table 1 Errors of the eigenvalues resulting from the present method (first example)

	Eigenvalue 1		Eigenvalue 2		Eigenvalue 3		Eigenvalue 4	
Iteration	Real	Imaginary	Real	Imaginary	Real	Imaginary	Real	Imaginary
0	2.1512552	0.7797515	1.6283808	0.6336569	2.5893398	0.9139061	3.4783698	1.1405928
1	0.0376853	0.0194171	0.0740226	0.0386519	0.1735704	0.0908992	0.7918929	0.3585053
2	0.0009842	0.0005183	0.0094122	0.0049531	0.0281767	0.0153288	0.3999646	0.1909331
3	0.0000295	0.0000162	0.0015202	0.0008250	0.0085762	0.0046863	0.2529920	0.1248303
4	0.0000008	0.0000005	0.0002289	0.0001341	0.0029887	0.0017053	0.1596758	0.0826897
5	0.0000001	0.0000000	0.0000305	0.0000206	0.0010016	0.0006262	0.0971780	0.0536903
6	0.0000001	0.0000000	0.0000034	0.0000030	0.0003061	0.0002237	0.0567344	0.0341650
7	0.0000001	0.0000000	0.0000002	0.0000004	0.0000816	0.0000776	0.0316483	0.0214215
8	0.0000001	0.0000000	−0.0000001	0.0000001	0.0000168	0.0000263	0.0166838	0.0133019
9	0.0000001	0.0000000	−0.0000001	0.0000000	0.0000011	0.0000088	0.0080867	0.0082104
10	0.0000001	0.0000000	−0.0000001	0.0000000	−0.0000014	0.0000029	0.0033505	0.0050493

Table 2 Errors of the eigenvalues resulting from the methods in Refs. 1 and 17 (first example)

	Eigenvalue 1		Eigenvalue 2		Eigenvalue 3		Eigenvalue 4	
Iteration	Real	Imaginary	Real	Imaginary	Real	Imaginary	Real	Imaginary
0	2.1838096	0.7842803	1.7100118	0.6460477	2.7415020	0.9335364	3.7014501	1.1661395
1	0.0401281	0.0198662	0.0816702	0.0400288	0.2000310	0.0954278	0.8787399	0.3703498
2	0.0010520	0.0005277	0.0100792	0.0050451	0.0316698	0.0158046	0.4238226	0.1936800
3	−0.0001706	0.0001661	−0.0016805	0.0016146	0.0073571	0.0091645	0.2004699	0.1397557
4	0.0000087	0.0000939	−0.0007934	0.0007919	0.0041386	0.0062445	0.1254392	0.1085293
5	0.0000280	0.0000594	−0.0002532	0.0004676	0.0025620	0.0044749	0.0793301	0.0868355
6	0.0000240	0.0000392	−0.0000509	0.0003051	0.0017376	0.0032970	0.0549462	0.0712837
7	0.0000183	0.0000264	0.0000148	0.0002105	0.0012408	0.0024765	0.0400760	0.0594741
8	0.0000136	0.0000181	0.0000345	0.0001507	0.0009278	0.0018912	0.0309810	0.0503048
9	0.0000104	0.0000127	0.0000388	0.0001109	0.0007208	0.0014663	0.0247807	0.0430165
10	0.0000079	0.0000090	0.0000376	0.0000834	0.0005791	0.0011532	0.0204588	0.0371398

Table 3 Real parts of the complex frequencies (rad/s) resulting from the present method (second example)

Iteration	Frequency 1	Frequency 2	Frequency 3	Frequency 4	Frequency 5	Frequency 6	Frequency 7	Frequency 8	Frequency 9	Frequency 10
0	−0.**351**257	−0.**45**9752	−3.35444	−20.2196	−29.4092	−16.2522	−54.9616	−57.6414	−154.475	−61.6006
2	−0.**33428**4	−0.**454051**	−0.**32**2134	−**6.**71912	−10.3897	−7.67817	−**7.**86572	−**19.**8781	−21.5751	−37.6786
4	−0.**334283**	−0.**454051**	−0.**321728**	−**6.48**547	−**9.**77570	−**6.77**129	−**7.**50688	−**18.**1923	−**16.**3887	4.99675
6	−0.**334283**	−0.**454051**	−0.**321728**	−**6.46**571	−**9.76**763	−**6.72**992	−**7.46**009	−**12.**5756	−**17.**9246	−**17.**3053
8	−0.**334283**	−0.**454051**	−0.**321728**	−**6.46483**	−**9.76748**	−**6.72766**	−**7.49**449	−**13.**3257	−**17.8**943	−**16.4**704
10	−0.**334283**	−0.**454051**	−0.**321728**	−**6.46483**	−**9.76748**	−**6.72764**	−**7.49**304	−**12.9**869	−**17.88**87	−**16.4**175
12	−0.**334283**	−0.**454051**	−0.**321728**	−**6.46483**	−**9.76748**	−**6.72764**	−**7.49290**	−**12.95**00	−**17.887**5	−**16.4**105
Exact	−0.**334283**	−0.**454051**	−0.**321728**	−**6.46483**	−**9.76748**	−**6.72764**	−**7.49290**	−**12.9449**	−**17.8871**	−**16.4082**

For comparison purpose, the errors resulting from use of the methods in Refs. 1 and 17 are listed in Table 2. Although there is little difference between the errors of the initial approximations resulting from the two kinds of methods, the errors obtained from the present method reduce much more quickly than those from methods of Refs. 1 and 17.

The damping matrix in this example is proportional to the stiffness matrix because of the particular selection of the damping matrix. As mentioned in the Introduction, the dynamic condensation method for undamped systems can be used to solve the problem directly. Here, we want to show that the dynamic condensation approaches for nonclassically damped systems can also be applied to proportionally damped models and that the present method is more efficient and accurate than the approaches in Refs. 1 and 17 for proportionally damped models.

B. Floating Raft Isolation System

For the second example, the floating raft isolation system described in Ref. 20 is considered. The machines to be isolated are denoted by $m_1 = 100$ kg and $m_2 = 120$ kg. A and B are rectangular plates and denote the raft frame and base, respectively. Their lengths, widths, and thicknesses are 1.2, 0.8, and 0.02 m and 2.8, 0.8, and 0.04 m, respectively. Their modulus of elasticity is $2.0E11$ N/m^2 and mass density is 7800 kg/m^3. The two short sides of plate B are simply supported, and the two long sides are free. The four sides of plate A are all free, and $k_1 = 1.0E5$ N/m, $k_2 = 5.0E5$ N/m, $c_1 = 100\ \text{N} \cdot \text{s/m}^2$, and $c_2 = 200\ \text{N} \cdot \text{s/m}^2$. The raft frame and the base are discretized using the finite element method. They are divided into 24 and 14 rectangular elements, respectively. The definition of the elements, nodes, and the connections with other components may be found in Ref. 20. The isolation system has a total of 179 degrees of freedom under this discretization.

The degrees of freedom associated with the two machines, and the translational degrees of freedom at nodes 2, 4, 8, 9, 14, 22 in the raft and at nodes 9 and 14 in the base are selected as the master degrees of freedom when the dynamic condensation is applied. All of the frequencies of the reduced model in former 12 iterations are listed in Tables 3 and 4. The results obtained from the methods in Refs. 1 and 17 are listed in Tables 5 and 6 for comparison purpose. In Tables 3–6 boldface numbers denote the significant digits. Exact indicates the results obtained from the full model. Because the reduced model is derived from the full model, it is reasonable to consider the full model as exact. Again, the results show that the proposed method is efficient for nonclassically damped systems. Although the reduced model only has 10 degrees of freedom, which is about one 18th of the full model, it is a good representation of the full model in frequency range (0, 500) rad/s after 10 iterations.

Because the lower eigenvalues usually converge much faster than the higher, the 9th and 10th eigenvalues of the reduced model are considered here. The errors of these two eigenvalues computed for four cases are shown in Figs. 2 and 3, respectively. In Figs. 2 and 3, case A denotes the results obtained from the methods in Refs. 1

Table 4 Imaginary parts of the complex frequencies (rad/s) resulting from the present method (second example)

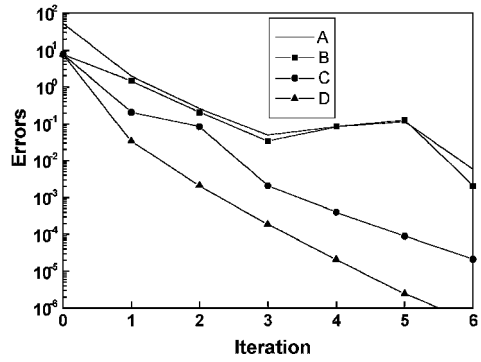
Iteration	Frequency 1	Frequency 2	Frequency 3	Frequency 4	Frequency 5	Frequency 6	Frequency 7	Frequency 8	Frequency 9	Frequency 10
0	**27**.6811	**30**.8131	193.028	362.290	510.015	673.639	732.313	1076.46	1922.65	2986.84
2	**27.4784**	**30.7270**	**67.4**832	**227**.711	**231**.199	**242**.250	**347**.848	555.324	565.686	1326.30
4	**27.4784**	**30.7270**	**67.4671**	**226**.812	**227.9**22	**238**.772	**340**.325	534.822	**542**.570	978.800
6	**27.4784**	**30.7270**	**67.4671**	**226.7**61	**227.902**	**238.6**49	**337**.011	**491**.904	**533.9**91	**546**.799
8	**27.4784**	**30.7270**	**67.4671**	**226.759**	**227.902**	**238.642**	**335.9**60	**420**.814	**533.9**40	**542**.968
10	**27.4784**	**30.7270**	**67.4671**	**226.759**	**227.902**	**238.642**	**335.931**	**418**.423	**533.935**	**542.8**39
12	**27.4784**	**30.7270**	**67.4671**	**226.759**	**227.902**	**238.642**	**335.931**	**418.3**27	**533.934**	**542.82**8
Exact	**27.4784**	**30.7270**	**67.4671**	**226.759**	**227.902**	**238.642**	**335.931**	**418.319**	**533.934**	**542.827**

Table 5 Real parts of the complex frequencies (rad/s) resulted from methods of Refs. 1 and 17 (second example)

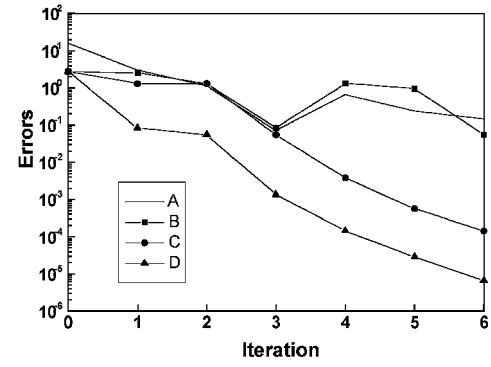
Iteration	Frequency 1	Frequency 2	Frequency 3	Frequency 4	Frequency 5	Frequency 6	Frequency 7	Frequency 8	Frequency 9	Frequency 10
0	−0.**3**52946	−0.**4**60291	−42.6293	−245.045	−281.990	−165.917	−617.559	−589.828	−978.437	−277.549
2	−0.**334284**	−0.**454051**	−0.**32**2140	**−6**.84264	−10.6025	−7.83154	−8.16662	**−1**9.2819	−22.4687	−34.6759
4	−0.**334283**	−0.**454051**	−0.**321**539	**−6**.50874	**−9.7**6172	**−6.7**6654	**−7**.52963	**−1**8.0864	−16.3650	5.53147
6	−0.**334283**	−0.**454051**	−0.**321**560	**−6.4**7890	**−9.7**5627	**−6.7**3457	**−7.4**2061	−13.0259	**−17**.9903	**−1**8.7647
8	−0.**334283**	−0.**454051**	−0.**321**574	**−6.4**7562	**−9.7**5746	**−6.7**3192	**−7.4**3290	−13.0361	**−17**.9497	**−16**.6137
10	−0.**334283**	−0.**454051**	−0.**321**587	**−6.4**7423	**−9.7**5834	**−6.7**3122	**−7.4**4992	−13.0567	**−17**.9439	**−16**.5342
12	−0.**334283**	−0.**454051**	−0.**321**597	**−6.4**7318	**−9.7**5889	**−6.7**3066	**−7.4**5960	−13.0879	**−17**.9428	**−16**.5134
Exact	−0.**334283**	−0.**454051**	−0.**321728**	**−6.46483**	**−9.76748**	**−6.72764**	**−7.49290**	**−12.9449**	**−17.8871**	**−16.4082**

Table 6 Imaginary parts of the complex frequencies (rad/s) resulted from methods of Refs. 1 and 17 (second example)

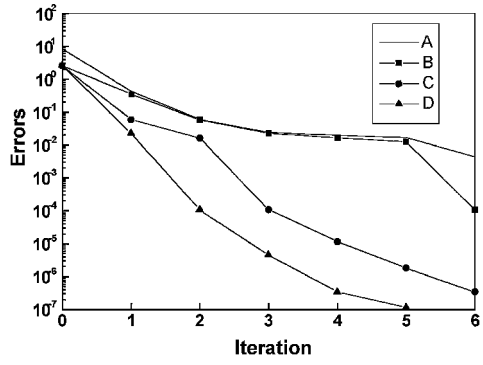
Iteration	Frequency 1	Frequency 2	Frequency 3	Frequency 4	Frequency 5	Frequency 6	Frequency 7	Frequency 8	Frequency 9	Frequency 10
0	**27**.7031	**30**.8217	648.039	1301.90	1783.21	2147.30	2568.93	3541.74	4999.02	6493.01
2	**27.4784**	**30.7270**	**67.4**935	**228**.193	**233**.059	**242**.776	**355**.506	559.382	**566**.196	1291.61
4	**27.4784**	**30.7270**	**67.46**89	**226**.932	**228**.368	**238**.978	**343**.999	537.682	**544**.449	998.693
6	**27.4784**	**30.7270**	**67.46**88	**226**.842	**228**.288	**238**.850	**339**.298	524.540	**536**.245	**554**.278
8	**27.4784**	**30.7270**	**67.46**88	**226**.831	**228**.262	**238**.828	**337**.788	**445**.774	**535**.780	**544**.407
10	**27.4784**	**30.7270**	**67.46**88	**226**.825	**228**.245	**238**.817	**337**.463	**436**.950	**535**.571	**544**.096
12	**27.4784**	**30.7270**	**67.46**88	**226**.821	**228**.230	**238**.809	**337**.271	**433**.454	**535**.445	**543**.985
Exact	**27.4784**	**30.7270**	**67.4671**	**226.759**	**227.902**	**238.642**	**335.931**	**418.319**	**533.934**	**542.827**



Real parts

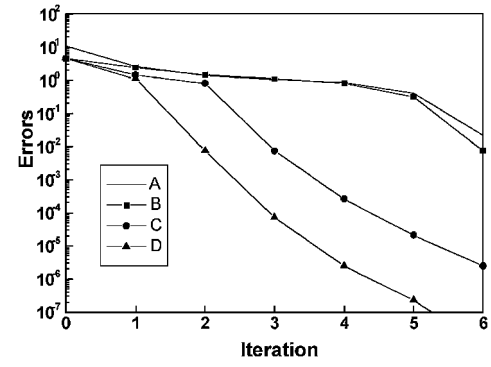


Real parts



Imaginary parts

Fig. 2 Errors of the ninth eigenvalue for the four cases.



Imaginary parts

Fig. 3 Errors of the 10th eigenvalue for the four cases.

and 17. Cases B, C, and D result from the proposed method for $k = 1, 2$, and 3, respectively. Clearly, the convergence of the present method is much faster than the methods in Refs. 1 and 17. It becomes faster with increasing k.

VI. Conclusions

Based on the subspace iteration method for nonclassically damped systems, an efficient approach for the dynamic condensation was derived. Two iterative schemes have been presented for this method. The proof of the convergence of these two schemes is very simple, whereas it is very difficult for the methods in Refs. 1 and 17. Two numerical examples were included to demonstrate the convergence of the present approach.

On one hand, the dynamic condensation matrix is independent of the system matrices and eigenproblem of the reduced model. Hence, it is unnecessary to calculate them at each of iterations as shown in iterative schemes 1 and 2. Significant computational effort, about $(k-1)(W2+W3)/k$, may be saved for each iteration. On the other hand, even though the classical iterative scheme is used in the two examples, the convergence of the present approach is much faster than with the methods in Refs. 1 and 17, especially when the approximate values of the reduced model are close to those of the full model. Therefore, the proposed method is much more computationally efficient than the methods in Refs. 1 and 17.

The reduced model can represent the full model in the low-frequency range after several iterations. Although several schemes for the selection of the master degrees of freedom can be used to accelerate the convergence, they do not work for many cases.[1] Furthermore, these schemes are usually very computationally expensive.

References

[1]Qu, Z.-Q., "Structural Dynamic Condensation: Theory and Application," Ph.D. Dissertation, State Key Lab. of Vibration, Shock and Noise, Shanghai Jiao Tong Univ., Shanghai, PRC, Oct. 1998.

[2]Guyan, R. J., "Reduction of Stiffness and Mass Matrices," *AIAA Journal*, Vol. 3, No. 2, 1965, p. 380.

[3]Irons, B., "Structural Eigenvalue Problems: Elimination of Unwanted Variables," *AIAA Journal*, Vol. 3, No. 5, 1965, pp. 961, 962.

[4]Miller, C. A., "Dynamic Reduction of Structural Models," *Journal of Structural Division*, Vol. 106, No. 10, 1980, pp. 2097–2108.

[5]O'Callahan, J. C., "A Procedure for an Improved Reduced System (IRS) Model," *Proceedings of the 7th International Modal Analysis Conference*, Society for Experimental Mechanics, Bethel, CT, 1989, pp. 17–21.

[6]Suarez, L. E., and Singh, M. P., "Dynamic Condensation Method for Structural Eigenvalue Analysis," *AIAA Journal*, Vol. 30, No. 4, 1992, pp. 1046–1054.

[7]Dubigeon, S., and Peseux, B., "Dynamic Condensation: the 'Free Mode' Method," *International Journal for Numerical Methods in Engineering*, Vol. 37, No. 20, 1994, pp. 3521–3543.

[8]Friswell, M. I., Garvey, S. D., and Penny, J. E. T., "Model Reduction Using Dynamic and Iterated IRS Technique," *Journal of Sound and Vibration*, Vol. 186, No. 2, 1995, pp. 311–323.

[9]Kim, K.-O., "Hybrid Dynamic Condensation for Eigenproblems," *Computers and Structures*, Vol. 56, No. 1, 1995, pp. 105–112.

[10]Friswell, M. I., Garvey, S. D., and Penny, J. E. T., "Using Iterative IRS Model Reduction Techniques to Calculate Eigenproblems," *Proceedings of the 15th International Modal Analysis Conference*, Society for Experimental Mechanics, Bethel, CT, 1997, pp. 1537–1543.

[11]Qu, Z., and Fu, Z., "An Iterative Method for Dynamic Condensation of Finite Element Models, Part I: Basic Method," *Journal of Shanghai Jiao Tong University*, English ed., Vol. 3, No. 1, 1998, pp. 83–88.

[12]Qu, Z.-Q., and Fu, Z.-F., "New Structural Dynamic Condensation Method for Finite Element Models," *AIAA Journal*, Vol. 36, No. 7, 1998, pp. 1320–1324.

[13]Qu, Z.-Q., "A Multi-Step Method for Matrix Condensation of Finite Element Models," *Journal of Sound and Vibration*, Vol. 214, No. 5, 1998, pp. 965–971.

[14]Kim, K.-O., and Kang, M.-K., "Convergence Acceleration of Iterative Modal Reduction Methods," *AIAA Journal*, Vol. 39, No. 1, 2001, pp. 134–140.

[15]Qu, Z.-Q., and Selvam, R. P., "Two-Step Methods for Dynamic Condensation," *Proceedings of the 42nd AIAA/ASME/ASCE/ASC Structures, Structural Dynamics, and Materials Conference* [CD-ROM], AIAA, Reston, VA, 2001.

[16]Ye, Y., and Qu, Z.-Q., "Iterative Methods for Dynamic Condensation of Structural Model," *Proceedings of the 18th International Modal Analysis Conference*, Society for Experimental Mechanics, Bethel, CT, 2000, pp. 1776–1782.

[17]Rivera, M. A., Singh, M. P., and Suarez, L. E., "Dynamic Condensation Approach for Nonclassically Damped Structures," *AIAA Journal*, Vol. 37, No. 5, 1999, pp. 564–571.

[18]Bathe, K. J., and Wilson, E. L., "Large Eigenvalue Problems in Dynamic Analysis," *Journal of Engineering Mechanics Division*, Vol. 98, No. EM6, 1972, pp. 1471–1485.

[19]Bathe, K. J., *Finite Element Procedures*, Prentice–Hall, Upper Saddle River, NJ, 1996, pp. 887–978.

[20]Qu, Z.-Q., and Selvam, R. P., "Dynamic Superelement Modeling Method for Compound Dynamic Systems," *AIAA Journal*, Vol. 38, No. 6, 2000, pp. 1078–1083.

S. Saigal
Associate Editor